

# PROPERTIES OF BÖRÖCZKY TILINGS IN HIGH DIMENSIONAL HYPERBOLIC SPACES

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ABSTRACT. In this paper we consider families of Böröczky tilings in hyperbolic space in arbitrary dimension, study some basic properties and classify all possible symmetries. In particular, it is shown that these tilings are non-crystallographic, and that there are uncountably many tilings with a fixed prototile.

## 1. Introduction

Let  $\mathbb{X}$  be either a Euclidean space, or a hyperbolic space, or a spherical space. Consider compact subsets  $T \subseteq \mathbb{X}$ , such that  $T$  is the closure of its interior. A collection of such sets  $\{T_1, T_2, \dots\}$  is called a *tiling*, if the union of the  $T_i$  is the whole space  $\mathbb{X}$  and the  $T_i$  do not overlap, i.e., the interiors of the tiles are pairwise disjoint. The  $T_i$  are called *tiles* of the tiling.

In 1975, K. Böröczky published some ingenious constructions of tilings in the hyperbolic plane  $\mathbb{H}^2$  [2]. His aim was to show that there is not such a natural definition of density in the hyperbolic plane  $\mathbb{H}^2$  as there is in the Euclidean plane  $\mathbb{E}^2$ . A very similar tiling is mentioned also in [10], wherefore these tilings are often attributed to R. Penrose. In context of a local theorem for regular point sets ([3], see also [4]), M.I. Shtogrin realized that the Böröczky tilings are not crystallographic. V.S. Makarov considered analogues of these tilings in  $n$ -dimensional hyperbolic space and paid attention to the fact that for all  $d \geq 2$ , Böröczky-type prototiles never admit isohedral tilings of  $\mathbb{H}^d$  [8]. This is of interest in the context of Hilbert's 18th problem. Let us give a simple description of one of Böröczky's constructions, which follows [6]. We assume familiarity with basic terms and facts of  $d$ -dimensional hyperbolic geometry, see for instance [1] (for  $d = 2$ , with emphasis of length and area in  $\mathbb{H}^2$ ) or [11] (for  $d \geq 2$ ).

Let  $\ell$  be a line in the hyperbolic plane of curvature  $\chi = -1$  (see Figure 1). Place points  $\{X_i \mid i \in \mathbb{Z}\}$  on  $\ell$ , such that the length of the line segment  $X_i X_{i+1}$  is  $\ln 2$  for all  $i \in \mathbb{Z}$ . Draw through every point  $X_i$  a horocycle  $E_i$  orthogonal to  $\ell$ , such that all the horocycles  $E_i$  have a common ideal point  $\mathcal{O}$  at infinity. On each  $E_i$ , choose points  $X_i^j$ , such that  $X_i^0 = X_i$  and the length of the arc  $X_i^j X_i^{j+1}$  is the same for all  $i, j \in \mathbb{Z}$ . Denote by  $\ell_j$  the line parallel to  $\ell$ , which intersects  $E_0$  at  $X_0^j$ , and intersects  $\ell$  at the ideal point  $\mathcal{O}$ . (In particular:  $\ell_0 = \ell$ .) Due to the choice  $|X_i X_{i+1}| = \ln 2$  and  $\chi = -1$ , the arc on  $E_1$  between  $\ell_0$  and  $\ell_1$  is twice the length as the arc on  $E_0$  between the same lines.

Let  $H_0$  be a strip between horocycle  $E_0$  and horocycle  $E_1$  (including  $E_0$  and  $E_1$  themselves), and  $L_i$  a strip between two parallel lines  $\ell_i$  and  $\ell_{i+1}$  (including  $\ell_i$  and  $\ell_{i+1}$  themselves). All intersections  $B_i := H_0 \cap L_i$ ,  $i \in \mathbb{Z}$ , are pairwise congruent shapes.

These shapes tile the plane [2],[6]. Indeed, they pave a horocyclic strip between horocycles  $E_0$  and  $E_1$ . It is clear that shape  $B_1$  can be obtained from  $B_0$  by means of a horocyclic turn

$\tau$  about the ideal point  $\mathcal{O}$  that moves point  $X_0 \in E_0$  to point  $X_0^1 \in E_0$ . Any shape  $B_i$  in the horocyclic strip  $H_0$  is obtained as  $\tau^i(B_0)$ . Now, let  $\lambda$  be a shift along  $\ell_0$  moving the strip  $H_0$  to the strip  $H_1$ . This shift transfers the pavement of  $H_0$  to a pavement of  $H_1$ . Thus, a sequence of shifts  $\lambda^j$  ( $j \in \mathbb{Z}$ ) along a line  $\ell_0$  extends the pavement of the horocyclic strip  $H_0$  to the entire plane, resulting in one of several possible Böröczky tilings.

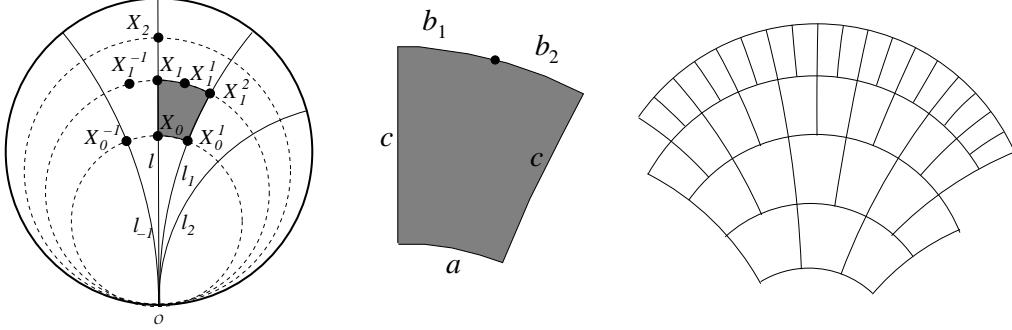


FIGURE 1. The construction of the prototile in the Poincaré-disc model (left), a Böröczky pentagon  $B$  with the types of the five edges indicated (centre), and a small part of a Böröczky tiling (right).

Let  $B := B_0 = L \cap H_0$ . The boundary of  $B$  consists of two straight line edges, labelled  $c$  in Figure 1 (centre), and two arcs, a shorter one labelled  $a$ , and a longer arc labelled  $b_1, b_2$ . The arcs are parts of horocycles. The longer arc is subdivided in two halves  $b_1$  and  $b_2$ . For brevity, we will say that  $B$  has three (horocyclic) edges  $a, b_1, b_2$  and two straight line edges  $c$ . Thus we regard  $B$  as a pentagon. Indeed, though the Böröczky pentagon  $B$  is neither bounded by straight line segments, nor it is a convex polygon, it suffices to replace all three horocyclic arcs by straight line edges in order to get a convex polygonal tile. But there is no need to use this modification here.

On the  $c$ -edges, we introduce an orientation from edge  $a$  to edge  $b_1$  or  $b_2$  respectively. An edge-to-edge tiling is a tiling in which any non-empty intersection of tiles is either their common edge or a vertex. An edge-to-edge tiling of the hyperbolic plane by Böröczky pentagons, respecting the orientations of  $c$ -edges, is called a *Böröczky tiling*, or, for the sake of brevity, *B-tiling*.

A tiling is called *crystallographic*, if the symmetry group of the tiling has compact fundamental domain, see [9]<sup>1</sup>. As we have already mentioned, all B-tilings are not crystallographic. In the next sections we will examine the symmetry group of B-tilings in detail.

In Section 2, we describe shortly the situation in the hyperbolic plane  $\mathbb{H}^2$ . Some basic terms and tools are introduced. Since Section 2 is a special case of Section 4, some proofs are omitted to Section 4. In Section 3 we state a necessary and sufficient condition for a tiling to be crystallographic [4]. This statement can be applied to B-tilings in  $\mathbb{H}^2$ . In Section 4 we consider Böröczky's construction in arbitrary dimension  $d \geq 2$  and obtain some results on  $B$ -tilings in  $\mathbb{H}^d$ . Section 5 uses these results and gives a complete classification of the symmetry groups of these tilings.

<sup>1</sup>Sometimes, a group is called crystallographic if its fundamental domain has finite volume [12].

We denote  $d$ -dimensional hyperbolic space by  $\mathbb{H}^d$ ,  $d$ -dimensional Euclidean space by  $\mathbb{E}^d$ , and the set of positive integers by  $\mathbb{N}$ .

## 2. The Böröczky Tilings in $\mathbb{H}^2$

In a Böröczky tiling every tile is surrounded by adjacent tiles in essentially the same way: A  $c$ -edge meets always a  $c$ -edge of an adjacent tile. A  $b_1$ -edge ( $b_2$ -edge resp.) meets an  $a$ -edge of an adjacent tile. The  $a$ -edge of a tile always touches either the  $b_1$ -edge or the  $b_2$ -edge of an adjacent tile.

Therefore we consider in the following definitions two kinds of subsets of tiles in a B-tiling, which are connected in two different ways: either we move from tile to an adjacent tile only across  $c$ -edges, or only across  $a$ -,  $b_1$ -, or  $b_2$ -edges.

**Definition 2.1.** *A ring in a B-tiling is a set  $(T_i)_{i \in \mathbb{Z}}$  of tiles ( $T_i \neq T_j$  for  $i \neq j$ ), such that  $T_i$  and  $T_{i+1}$  share  $c$ -edges for all  $i \in \mathbb{Z}$  (see Figure 2).*

**Definition 2.2.** *Let  $I \subset \mathbb{Z}$  be a set of consecutive numbers. A horocyclic path is a sequence  $(T_i)_{i \in I}$  of tiles ( $T_i \neq T_j$  for  $i \neq j$ ), such that, for all  $i \in \mathbb{Z}$ ,  $T_i$  and  $T_{i+1}$  share an edge which is not a  $c$ -edge (see Figure 2).*

We always require rings and horocyclic paths to contain no loops, i.e., all tiles in the sequence are pairwise different. A ring is a sequence of tiles which is infinite in both directions, shortly *biinfinite*. A ring forms a pavement of a horocyclic strip bounded by two consecutive horocycles  $E_i$  and  $E_{i+1}$ . A horocyclic path is either finite, or infinite in one direction, or biinfinite.

If, in the case of horocyclic paths, an  $a$ -edge of  $T_i$  touches a  $b_1$ - or  $b_2$ -edge of  $T_{i+1}$ , we say shortly: At this position the horocyclic path goes *down*, otherwise we say it goes *up*.

**Proposition 2.3.** *Any horocyclic path in a B-tiling contains either only ups, or only downs, or is of the form down-down- · · · -down-up-up- · · · -up.*

*Proof.* There is only one way to pass from a tile through its  $a$ -edge to an adjacent tile. So there is only one way to go down from a given tile. If we have in a horocyclic path the situation 'up to  $T_i$ , down to  $T_{i+1}$ , up to  $T_{i+2}$ ', it follows  $T_i = T_{i+2}$ . This situation is ruled out by the requirement, that all tiles in a horocyclic path are different. So 'up-down' cannot happen, which leaves only the possibilities mentioned in Proposition 2.3.  $\square$

There are always two possibilities to go up from some tile  $T$  in a B-tiling. That means, there are  $2^k$  different paths starting in  $T$  and containing exactly  $k$  ups (and no downs). Since there is only one possibility to go down from any tile, there is only one infinite horocyclic path of the form down-down-down- · · · starting in  $T$ .

**Definition 2.4.** *The unique infinite path of the form down-down- · · · starting at a tile  $T$  is called the tail (of  $T$ ).*

If tiles  $T$  and  $T'$  and also tiles  $T''$  and  $T'''$  are linked by horocyclic paths  $w$  and  $w'$  respectively, then the tiles  $T$  and  $T'''$  are also linked by a horocyclic path. Therefore, the property of tiles to be linked by a horocyclic path is an equivalence relation. Hence a set of tiles is partitioned into a number of non-overlapping classes  $\mathcal{P}_1, \mathcal{P}_2, \dots$  called pools.

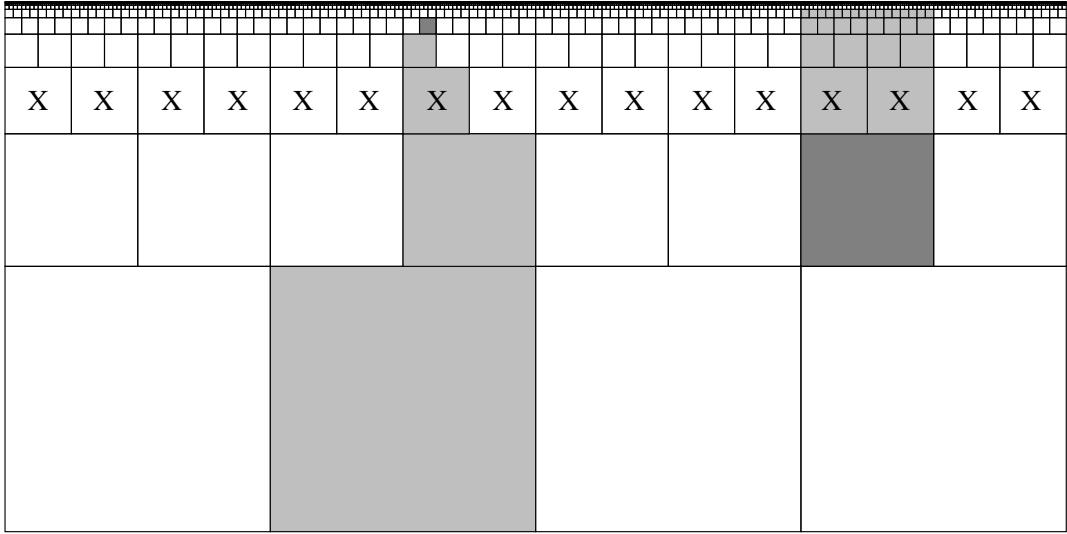


FIGURE 2. A patch of the Böröczky tiling in the half-plane model. The tiles marked by an X show (a part of) a ring. The grey shaded tiles show: a horocyclic path of length five (left, it is also the beginning of the tail of the dark tile on top); the tower on the dark tile (right, for the definition of a tower, cf. Section 4).

**Definition 2.5.** *The set of all tiles which can be linked with some given tile by a horocyclic path forms a pool.*

It is clear that if  $T \in \mathcal{P}$  then  $t(T) \subset \mathcal{P}$ .

**Definition 2.6.** *Two tails  $t(T) = (T = T_0, T_1, \dots)$ ,  $t(T') = (T' = T'_0, T'_1, \dots)$  in a B-tiling are called cofinally equivalent, written  $t(T) \sim t(T')$ , if there are  $m, n \in \mathbb{N}$  such that  $T_{m+i} = T'_{n+i}$  for all  $i \in \mathbb{N}$ .*

If  $t(T) \sim t(T')$  then  $T$  and  $T'$  are linked by a horocyclic path

$$T = T_0, T_1, \dots, T_m, T'_{n-1}, \dots, T'_1, T'_0 = T',$$

hence  $T$  and  $T'$  belong to one pool. The inverse statement is also obviously true. Thus, the following is true.

**Proposition 2.7.** *Tiles  $T$  and  $T'$  belong to a pool if and only if their tails are cofinally equivalent.*

A tail  $(T_0, T_1, T_2, \dots)$  gives rise to a sequence  $s(T) = (s_1, s_2, \dots) \in \{-1, 1\}^{\mathbb{N}}$  in the following way: If the  $a$ -edge  $a$  of  $T_{i-1}$  coincides with the  $b_1$ -edge of  $T_i$ , then set  $s_i := 1$ , otherwise (if it coincides with the  $b_2$ -edge) set  $s_i := -1$ .

**Definition 2.8.** *Let  $s = (s_1, s_2, \dots), s' = (s'_1, s'_2, \dots) \in \{-1, 1\}^{\mathbb{N}}$ .*

- *$s$  and  $s'$  are cofinally equivalent, denoted by  $s \sim s'$ , if there exist  $m, n \in \mathbb{N}$ , such that  $s_{m+i} = s'_{n+i}$  for all  $i \in \mathbb{N}$ .*
- *$s$  is periodic (with period  $k$ ), if there exists  $k \geq 2$  such that  $s = (s_k, s_{k+1}, \dots)$ .*

- $s$  is cofinally periodic, if  $s$  is cofinally equivalent to a periodic sequence.

**Proposition 2.9.** *Every pool contains infinitely many tiles. Moreover, every intersection of any ring with any pool contains infinitely many tiles.*

*Proof.* The tail of any tile  $T$  contains infinitely many tiles, all belonging to the same pool. Moreover, from any tile there are starting  $2^k$  horocyclic paths of length  $k$  going upwards. The final tiles of these paths are all in the same ring.

So, by going down from  $T$  by  $k$  steps and after that going up  $k$  steps gives us horocyclic paths to  $2^k$  different tiles in the same ring as  $T$  and in the same pool as  $T$ . Since  $k$  can be chosen arbitrary large, the claim follows.  $\square$

**Proposition 2.10.** *In a B-tiling in  $\mathbb{H}^2$  there is either one pool or two pools.*

*Proof.* First of all, note the following. Let  $x$  be an interior point of some pool, and let  $\ell'$  be a line such that  $x \in \ell'$ , and such that  $\ell'$  contains the ideal point  $\mathcal{O}$  at infinity (compare Section 1). Then  $\ell'$  is contained entirely in the interior of the pool. Therefore, if a point  $x$  is on the boundary of two pools then the line  $\ell'$  entirely belongs to the boundary of two pools.

Assume there is more than one pool. Consider some ring  $R$ . Each of the pools has infinite intersection with  $R$ . Let  $T$  and  $T' \in R$  be adjacent tiles from different pools, say,  $\mathcal{P}_1$  and  $\mathcal{P}_2$ . Then a line  $\ell'$  containing the common  $c$ -edge of  $\mathcal{P}_1$  and  $\mathcal{P}_2$  separates these pools. Moreover, since a pool is a linearly connected set, these pools lie on opposite sides each of the line  $\ell'$ . Assume there is a third pool  $\mathcal{P}_3$ . Let it lie on the same side of  $\ell'$  as  $\mathcal{P}_2$ . Since  $\mathcal{P}_3 \cap R \neq \emptyset$ , there is some tile  $T_3 \in \mathcal{P}_3 \cap R$ . There exists a line  $\ell''$  which separates  $\mathcal{P}_2$  and  $\mathcal{P}_3$ . The pool has to lie between two parallel lines  $\ell'$  and  $\ell''$ . But this impossible because in this case the intersection  $\mathcal{P} \cap R$  is finite what contradicts Proposition 2.9.  $\square$

In particular, there are two pools if and only if there are tiles  $T, T' \in \mathcal{T}$ , whose tails have sequences  $s(T) = (1, 1, 1, \dots)$  and  $s(T') = (-1, -1, -1, \dots)$ .

**Proposition 2.11.** *In any B-tiling holds:  $t(T) \sim t(T') \Leftrightarrow s(T) \sim s(T')$*

*Proof.* One direction ( $\Rightarrow$ ) is clear from the construction of  $s(T)$  out of  $t(T)$ .

The other direction: If there is only one pool, all tiles are cofinally equivalent, and we are done. If there are two pools  $\mathcal{P}_1 \neq \mathcal{P}_2$ , we know that these pools corresponds to sequences  $(-1, -1, -1, \dots)$  and  $(1, 1, 1, \dots)$ . Therefore, if  $s(T) \sim s(T')$ , then  $T$  and  $T'$  belong to the same pool:  $t(T) \sim t(T')$ .  $\square$

Let  $\text{Sym}(\mathcal{T})$  be the symmetry group of  $\mathcal{T}$ , i.e., the set of all isometries  $\varphi$  where  $\varphi(\mathcal{T}) = \mathcal{T}$ . The group of order two is denoted by  $\mathcal{C}_2$ .

**Theorem 2.12.** *The symmetry group  $\text{Sym}(\mathcal{T})$  of any B-tiling in  $\mathbb{H}^2$  is*

- isomorphic to  $\mathbb{Z} \times \mathcal{C}_2$  in the case of two pools,
- isomorphic to  $\mathbb{Z}$  in the case of one pool and  $s(T)$  periodic for some  $T$ ,
- trivial else.

*Proof.* (in Section 4)  $\square$

If there is one pool and  $s(T)$  is periodic for some  $T$ , it can happen, that  $\text{Sym}(\mathcal{T})$  contains shifts along a line, and also 'glide-reflections', i.e. a reflection followed by a shift along a line. The latter is the case, if  $s = (s_1, s_2, \dots)$  has a period  $k$  as in Definition 2.8, where  $k$  is an even number, and if holds:

$$(1) \quad s = (-s_{k/2+1}, -s_{k/2+2}, \dots).$$

In the following, we mean by *essential period*  $\frac{k}{2}$ , if (1) holds, otherwise  $k$ .

*Example:* A tiling  $\mathcal{T}$  with a periodic sequence

$$s(T) = (1, 1, -1, -1, 1, 1, -1, -1, \dots)$$

has period 4, and essential period 2.  $\mathcal{T}$  has a symmetry  $\varphi$ , where  $\varphi$  is a glide-reflection. The action of the reflection on  $s(T)$  gives  $(-1, -1, 1, 1, -1, -1, 1, 1, \dots)$ . This is followed by a shift along an edge of type  $C$  along two tiles which gives

$$(1, 1, -1, -1, 1, 1, -1, -1, \dots) = s(T).$$

**Corollary 2.13.** *The fundamental domain of  $\text{Sym}(\mathcal{T})$  is*

- one half ring in the case of two pools,
- the union of  $k$  rings in the case of one pool and  $s(T)$  for some  $T$  periodic, where  $k$  is the essential period of  $s(T)$ ,
- $\mathbb{H}^2$  else.

### 3. Böröczky tilings are non-crystallographic

Let  $\mathcal{T}$  be a face-to-face tiling in  $\mathbb{R}^d$  or  $\mathbb{H}^d$ .

**Definition 3.1.** *The 0-corona  $C_0(T)$  of a tile  $T$  is  $T$  itself. The  $k$ -corona  $C_k(T)$  of  $T$  is the complex of all tiles of  $\mathcal{T}$  which have a common  $(d-1)$ -face with some  $T' \in C_{k-1}(T)$ .*

Figure 3 shows a tile  $T$  in a B-tiling together with its first and second coronae. Note that, for  $T \neq T'$ , the coronae  $C_k(T)$  and  $C_k(T')$  can coincide as complexes. Nevertheless, the corona denoted by  $C_k(T)$  is considered as a corona about the centre  $T$ , whereas corona  $C_k(T')$  considered as a corona about  $T'$ .

**Definition 3.2.** *Coronae  $C_k(T)$  and  $C_k(T')$  are considered as congruent if there is an isometry that moves  $T$  to  $T'$  and  $C_k(T)$  to  $C_k(T')$ .*

Given a tiling  $\mathcal{T}$ , denote by  $N_k$  the number of congruence classes of  $k$ -coronae.

**Proposition 3.3.** *The number  $N_k$  of different  $k$ -coronae ( $k \geq 1$ ) in a B-tiling is  $2^{k-1}$ , up to isometries.*

*Proof.* Let us enumerate all rings in a biinfinite way. Given a tile  $T$ , we denote the ring containing  $T$  by  $\mathcal{R}_0$ . All the rings lying upward from  $\mathcal{R}_0$  are enumerated as  $\mathcal{R}_{-1}, \mathcal{R}_{-2}, \mathcal{R}_{-3}, \dots$ . All rings going downward from  $\mathcal{R}_0$  are enumerated by positive integers  $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \dots$

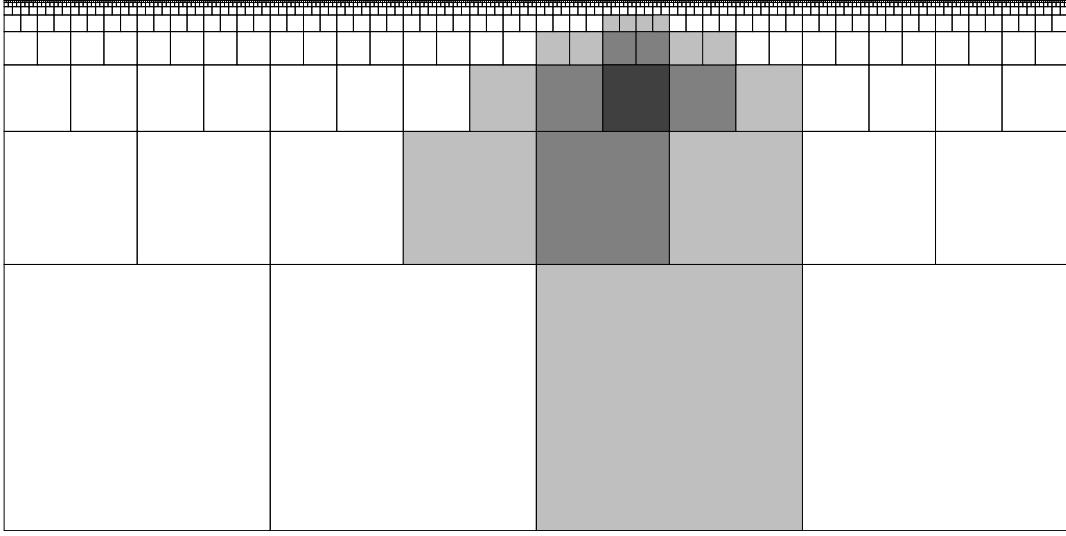


FIGURE 3. The first corona (medium grey) and the second corona (all grey tiles) of the dark tile.

It is easy to see that  $C_k(T) \cap \mathcal{R}_i \neq \emptyset$  if and only if  $-k \leq i \leq k$ . It is also easy to check that for any tiles  $T$  and  $T' \in \mathcal{R}_0$  the following complexes in  $\mathcal{T}$

$$C_k^-(T) := C_k(T) \cap \left( \bigcup_{i=-k}^0 \mathcal{R}_i \right) \quad \text{and} \quad C_k^-(T') := C_k(T') \cap \left( \bigcup_{i=-k}^0 \mathcal{R}_i \right)$$

are pairwise congruent. In other words: All  $k$ -coronae look the same, if we only look at the rings  $\mathcal{R}_0, \mathcal{R}_{-1}, \mathcal{R}_{-2}, \dots$  (i.e., the tiles 'above'  $T$ ). Moreover, since  $T$  and  $C_k^-(T)$  are both mirror symmetric, this congruence can be realized by two isometries both moving  $T$  into  $T'$ . In addition, the same is true also for any two tiles from a B-tiling.

The upper part of any corona is on one convex side of a horocycle. This part obviously contains a half plane cut off by a line perpendicular to  $\ell$ . Thus, all the tiles in  $\mathcal{T}$  for any  $k \in \mathbb{N}$  have  $k$ -coronae whose larger parts are pairwise congruent.

So, the upper part  $C_k^-(T)$  of some  $k$ -corona  $C_k(T)$  around  $T$  is uniquely determined independently of  $T$ . The entire corona  $C_k(T)$  is determined completely (up to orientation) by the first  $k$  members of the tail  $t(T)$ , or, what is equivalent, by the first numbers  $s_1, s_2, \dots, s_k$  in the sequence  $s(T)$ . Our next task is to show that in a given tiling for any finite sequence of  $k \pm 1$ 's there is a tile  $T$  whose tail is encoded exactly by this sequence.

Let us enumerate the tiles in ring  $\mathcal{R}_0$  (the ring containing  $T$ ) by upper indices. Let  $T^0 := T$  and let other tiles on one side of  $T^0$  be enumerated by positive  $i$ 's and on the other side by negative  $i$ 's in a consecutive manner.

Let  $s_k(T^0) = (s_1, s_2, \dots, s_k)$ . Then (see Figure 2) all tiles  $T^{2n}$  from ring  $\mathcal{R}_0$  have the same first number  $s_1$  in  $s(T^{2n})$ . And, in general, it is easy to see that every  $T^i \in \mathcal{R}_0$ , where  $i \equiv 0 \pmod{2^k}$ , has the same first  $k$  numbers in  $s(T^i)$ . It follows that in a biinfinite sequence  $(T_i) \in \mathcal{R}_0$  each  $2^k$ -th tile has the same coronae encoded by  $s_k(T^0)$ .

Consider in ring  $\mathcal{R}_0$  a segment  $T^0, T^1, T^2, \dots, T^{2^k-1}$ . Recall that  $s_k(T^0) = (s_1, s_2, \dots, s_k)$ . Now, one can easily check that either  $s_k(T^1) = (-s_1, s_2, s_3, \dots, s_k)$ , or  $s_k(T^2) = (s_1, -s_2, s_3, \dots, s_k)$ , compare for instance Figure 2.

And, in general, take  $0 \leq i \leq 2^k - 1$ . The number  $i$  can be represented in a unique way as

$$i = \sum_{j=0}^{k-1} \delta_j 2^j,$$

where  $\delta_j = 0$  or  $1$ . Then, as one can easily see,

$$s_k(T^i) = \{(-1)^{\delta_j} s_j \mid 0 \leq j \leq k\}.$$

Thus, any B-tiling contains  $2^k$  different  $k$ -coronae, up to orientation.  $\square$

Now we should emphasise that the number  $N_k$  of coronae in a B-tiling is unbounded, as  $k$  tends to infinity. But in any crystallographic tiling (independently on how it is defined, as with either compact or cocompact fundamental domain), the number of coronae classes is always bounded:  $N_k \leq m < \infty$ , where  $m$  is some fixed number independent of  $k$ . Therefore we have obtained the following result.

**Theorem 3.4.** *All B-tilings are non-crystallographic.*

On the other side, this last theorem (in the context of Proposition 3.3 and a local theory on crystallographic structures) can be considered as a consequence of the following theorem, which appears in [4]. This theorem is a generalization of the Local Theorem, see [3].

**Theorem 3.5.** *Let  $\mathbb{X}^d$  be a Euclidean or hyperbolic space. A tiling  $\mathcal{T}$  in  $\mathbb{X}^d$  is crystallographic if and only if the following two conditions hold for some  $k \geq 0$ :*

- (1) *For the numbers  $N_k$  of  $k$ -coronae in  $\mathcal{T}$  holds:  $N_{k+1} = N_k$ , and  $N_k$  is finite.*
- (2)  *$S_{k+1}(i) = S_k(i)$  for  $1 \leq i \leq N_k$ ,*

where  $S_k(i)$  denotes the symmetry group of the  $i$ -th  $k$ -corona.

Note that Condition (2) in the theorem makes sense only when Condition (1) is fulfilled. In a B-tiling, condition (2) is violated for  $k = 0$ , and condition (1) is violated for all  $k \geq 1$ . Thus Theorem 3.4 is a particular case of the Local Theorem.

But in fact, the necessity part of the Local Theorem is trivial in contrast to a valuable sufficiency part. Indeed, if for any  $k \in \mathbb{N}$  at least one condition does not hold, this means that  $N_k \rightarrow \infty$  as  $k \rightarrow \infty$ . The latter one implies that the tiling under consideration is non-crystallographic.

#### 4. Böröczky Tilings in $\mathbb{H}^{d+1}$

Let us give the construction of a  $d + 1$ -dimensional prototile. The construction in Section 1 is a special case of this one, with  $d = 1$ . Throughout this section, we will make strong use of the fact that any  $d$ -dimensional horosphere  $E$  in  $\mathbb{H}^{d+1}$  is isometric to the Euclidean space  $\mathbb{E}^d$ , see for instance [11, §4.7]. Whenever we consider a subset  $A$  of a horosphere  $E$ , we will freely switch between regarding  $A$  as a subset of  $\mathbb{E}^d$  and a subset of  $E$ .

Let  $\ell$  be a line in  $\mathbb{H}^{d+1}$ . Choose a horosphere  $E_0$  orthogonal to  $\ell$ . Let  $\square'$  be a  $d$ -dimensional cube in  $E_0$ , centred at  $E_0 \cap \ell$ . Let  $H_1, \dots, H_{2d}$  be hyperbolic hyperplanes, orthogonal to  $E_0$ , such that each  $H_i$  contains one of the  $2d$  distinct  $(d-1)$ -faces of the cube  $\square'$ . Denote by  $H_i^+$  the halfspace defined by  $H_i$  which contains the cube  $\square'$ , and let  $C_{d+1} := \bigcap_{i=1}^{2d} H_i^+$ . Note that the intersection  $C_{d+1} \cap E_0 = \square'$ . Let  $E_1$  be another horosphere, such that  $E_1$  is concentric with  $E_0$ ,  $E_1$  is not contained in the convex hull of  $E_0$ , and the distance of  $E_0$  and  $E_1$  is  $\ln 2$ . Then,  $C_{d+1} \cap E_1$  is a  $d$ -dimensional cube of edge-length 2. Without loss of generality, let this cube in  $E_1$  be  $\square := \{x = (x_1, \dots, x_d) \in E_1 \mid -1 \leq x_i \leq 1\}$ . Divide  $\square$  into  $2^d$  cubes

$$\square_{(\sigma_1, \sigma_2, \dots, \sigma_d)} = \{x \in \square \mid x_i \sigma_i \geq 0\}, \sigma_i \in \{-1, 1\}.$$

Let  $L_0$  be the layer between  $E_0$  and  $E_1$ . A *Böröczky prototile* in  $\mathbb{H}^{d+1}$  is defined as

$$B := C_{d+1} \cap L_0.$$

By construction, the prototile  $B$  has  $2^d + 2d + 1$  facets: one 'lower' facet (a unit Euclidean cube in  $E_0$ , denoted as  $a$ -facet),  $2^d$  'upper' facets (unit Euclidean cubes in  $E_1$ , denoted as  $b$ -facets), and  $2d$  aside facets, denoted as  $c$ -facets. One should mention that the Böröczky prototile is not a convex polyhedron. Moreover, it is not a polyhedron at all. Though we call them facets, the lower and upper facets are Euclidean  $d$ -cubes isometrically embedded into hyperbolic space and they do not lie in hyperbolic hyperplanes. The  $c$ -facets lie in hyperbolic hyperplanes, but for  $d \geq 2$  they are also not  $d$ -dimensional polyhedra, because their boundaries do not consist of hyperbolic polyhedra of dimension  $d-1$ . A  $c$ -facet of a  $d+1$ -dimensional Böröczky prototile is a point set which lies in a hyperbolic hyperplane  $H$  and is contained between the intersections  $E_0 \cap H$  and  $E_1 \cap H$ . Since the intersections are also horocycles of dimension  $d-1$ , the  $c$ -facets are Böröczky prototiles of dimension  $d-1$ .

**Proposition 4.1.** *Prototile  $B$  admits a face-to-face tiling  $T$  by its copies.*

*Proof.* Take a horosphere  $E_0$  and a unit cube  $\square' \subset E_0$ . There are parabolic turns  $g_i$  ( $i = 1, 2, \dots, d$ ) of  $\mathbb{H}^{d+1}$  such that their restrictions  $g_i|_{E_0}$  on  $E_0$  are translations of  $E_0$  along edges of  $\square'$ . The  $g_i$  ( $i = 1, 2, \dots, d$ ) span an Abelian group  $G$  isomorphic to  $\mathbb{Z}^d$ . The orbit of the Böröczky prototile under  $G$  is a pavement of the layer between  $E_0$  and  $E_1$  by copies of  $B$ .

Let  $\tau$  be a shift of  $\mathbb{H}^{d+1}$  along the line  $\ell$  moving  $E_0$  to  $E_1$ . Then, the shifts  $\tau^k$ ,  $k \in \mathbb{Z}$ , move the pavement of the mentioned layer into all other layers, resulting in a tiling of the whole space.  $\square$

We call any face-to-face tiling of  $\mathbb{H}^{d+1}$  with prototile  $B$  a *Böröczky tiling*, or shortly *B-tiling*. Analogously to  $\mathbb{H}^2$ , we call  $a$ - and  $b$ -facets of  $B$  *horospheric* facets. We should emphasise that if some horosphere  $E_0$  is fixed, then a unique sequence  $(E_i)_{i \in \mathbb{Z}}$  of horospheres orthogonal to  $\ell$  is induced, provided the distance along  $\ell$  between any two consecutive horospheres  $E_i$  and  $E_{i+1}$  is  $\ln 2$ . By construction of a B-tiling, all  $a$ - and  $b$ -facets induce a tiling of each horosphere by pairwise parallel unit cubes.

Let us state the definition of an analogue of a two-dimensional ring.

**Definition 4.2.** *Let  $T$  be a B-tiling in  $\mathbb{H}^{d+1}$ , and let  $L_i$  be a layer of space between horospheres  $E_i$  and  $E_{i+1}$ . The set  $\mathcal{R}_i$  of all tiles from  $T$  lying in  $L_i$  is called a layer (of the tiling).*

It is obvious that two tiles belong to one layer if and only if there is some path  $T_i$ ,  $i \in I$ , such that for any  $i, i+1 \in I$ , the tiles  $T_i$  and  $T_{i+1}$  have a  $c$ -facet in common.

Let  $I$  be a segment of  $\mathbb{Z}$  of consecutive numbers. In analogy to Section 2, a sequence  $(T_i)_{i \in I}$  of tiles is called a horospheric path, if for all  $i, i+1 \in I$  the tiles  $T_i$  and  $T_{i+1}$  share a horospheric facet, and  $T_i \neq T_j$  if  $i \neq j$ . By construction, the single  $a$ -facet of any tile in some Böröczky tiling in  $\mathbb{H}^{d+1}$  coincides with one of  $2^d$   $b$ -facets of an adjacent tile. To describe horospheric paths by sequences, we use the alphabet

$$\mathcal{A} = \{\sigma \mid \sigma = (\sigma_1, \sigma_2, \dots, \sigma_d), \sigma_i = \pm 1\}, |\mathcal{A}| = 2^d$$

First of all, we emphasise that for any tile  $B'$  from a B-tiling, a natural bijection between its  $2^d$   $b$ -facets and the elements of  $\mathcal{A}$  is uniquely determined, provided such a bijection (between  $b$ -facets and elements of  $\mathcal{A}$ ) is already established for some tile  $B$ . Indeed, assume that a tile  $B'$  lies in the  $i$ -th layer  $\mathcal{R}_i$ . Then the bijection for  $B$  is canonically carried by a single shift  $\tau^i$  of the tile  $B$  along  $\ell$  into  $\mathcal{R}_i$ , followed by an appropriate translation inside the layer  $\mathcal{R}_i$ .

Analogous to the two-dimensional case, any horospheric path  $(T_i)_{i \in I}$  of the form down-down- $\dots$  gives rise to a word  $s(T) = (\sigma^{(i)})_{i \in I}$  over  $\mathcal{A}$  in the following way: If an  $a$ -facet of  $T_i$  lies on a  $b$ -facet of  $T_{i+1}$ , which corresponds to the cube  $\square_{(\sigma_1, \dots, \sigma_d)}$ , we set  $\sigma^{(i)} := (\sigma_1, \dots, \sigma_d)$ .

As in Section 2, the tail  $t(T)$  of a tile  $T$  is the unique infinite horospheric path  $(T = T_0, T_1, \dots)$  beginning in  $T$  of the form down-down-down- $\dots$ .

The definition of a pool and of the equivalence of tails goes exactly as in Definition 2.5 and Definition 2.6 (replace 'horocyclic' by 'horospheric'). Moreover, the proofs of Propositions 2.3 and 2.9 work in any dimension, so both propositions are valid here.

**Definition 4.3.** *For a tile  $T \in \mathcal{T}$ , let  $W(T)$  be the set of all tiles  $T'$ , such that  $T$  can be linked with  $T'$  by a horospheric path  $T, \dots, T'$  of the form up-up- $\dots$ . We call  $W(T)$  a tower, or the tower on  $T$ .*

For an example of a tower in  $\mathbb{H}^2$ , see Figure 2.

**Proposition 4.4.** *Let  $\mathcal{P} \subset \mathcal{T}$  be a pool,  $T \in \mathcal{P}$  and  $t(T) = (T = T_0, T_1, T_2, \dots)$ . Then*

$$W(T_i) \subset W(T_{i+1}) \quad \text{and}$$

$$(2) \quad \mathcal{P} = \bigcup_{T' \in t(T)} W(T') = W(T_0) \cup W(T_1) \cup W(T_2) \cup \dots$$

*Proof.* Since a horospheric path  $T_{i+1}, T_i$  is of the form 'up', any tile  $T' \in W(T_i)$  can be linked with  $T_{i+1}$  by a path of the form up-up- $\dots$ , namely  $T_{i+1}, \dots, T'$ . This implies  $W(T_i) \subset W(T_{i+1})$ .

Of course, all tiles  $T' \in t(T)$  are in the same pool as  $T$  because they are connected by a horospheric patch in  $t(T)$ . All tiles in  $W(T')$  are in the same pool as  $T'$ , therefore in the same pool as  $T$ . This shows  $\mathcal{P} \supseteq \bigcup_{T' \in t(T)} W(T')$ .

Let  $T'' \in \mathcal{P}$ . Then, by definition of a pool, exists a horospheric path connecting  $T$  and  $T''$ . By Proposition 2.3, this path is either of the form up-up- $\dots$ , or down-down- $\dots$ , or down-down- $\dots$ -down-up- $\dots$ -up.

If this path  $(T, \dots, T'')$  is up-up- $\dots$ , then  $T'' \in W(T_i)$  for any  $T_i \in t(T)$ , thus  $T''$  is contained in the right hand side of (2).

If the path  $(T, \dots, T'')$  is down-down- $\dots$ , then  $T'' \in t(T)$ , thus  $T''$  is contained in the right hand side of (2).

If the path  $(T, \dots, T'')$  is down-down- $\dots$ -down-up- $\dots$ -up, then this path contains a tile  $\tilde{T}$ , such that the path  $(T, \dots, T'')$  consists of two paths: a down-down- $\dots$ -down path  $(T, \dots, \tilde{T})$ , and an up-up- $\dots$  path  $(\tilde{T}, \dots, T'')$ . From here it follows that  $\tilde{T} \in t(T)$  and  $T'' \in W(\tilde{T})$ , thus  $T''$  is contained in the right hand side of (2).  $\square$

Let us now investigate the structure of a tower and a pool, respectivley. Consider a pool  $\mathcal{P} \subseteq \mathcal{T}$ . Fix some horosphere  $E_0$  and let a tile  $T \in \mathcal{P}$  have its  $b$ -facets in  $E_0$ . Obviously, the intersection

$$u_0(T) := \text{supp}(W(T)) \cap E_0$$

is exactly the union of  $b$ -facets (which are unit cubes) of  $T$ . (Here,  $\text{supp}(W(T))$  denotes the support of  $W(T)$ , that is, the union of all tiles in  $W(T)$ ). This union is a  $d$ -dimensional cube  $C_2$  of edge length 2. We introduce in  $E_0$ , as in Euclidean  $d$ -space, a Cartesian coordinate system with axes parallel to the edges of  $C_2$ . Therefore, there are  $a_i^{(0)}, b_i^{(0)} \in \mathbb{R}$ , such that

$$u_0(T) = \{x \in E_0 \mid a_i^{(0)} \leq x_i \leq b_i^{(0)}\}$$

where  $b_i^{(0)} - a_i^{(0)} = 2$ . Consider the tail  $t(T) = (T = T_0, T_1, T_2, \dots)$  and let

$$u_0(T_j) := \text{supp}(W(T_j)) \cap E_0, \quad (T_j \in t(T_0))$$

Let us calculate a representation of  $u_0(T_{j+1})$  in terms of  $u_0(T_j)$  and the biinfinite word  $(\sigma_i)_{i \in I}$  encoding  $t(T)$ . The intersection  $u_0(T_j) = \text{supp}(W(T_j)) \cap E_0$  is a  $d$ -cube

$$u_0(T_j) = \{x \in E_0 \mid a_i^{(j)} \leq x_i \leq b_i^{(j)}\}$$

of edge-length  $2^{j+1}$ :  $b_i^{(j)} - a_i^{(j)} = 2^{j+1}$ . Now we calculate coordinates of the section  $u_0(T_{j+1})$ . Since  $W(T_j) \subseteq W(T_{j+1})$ , we have  $a_i^{(j+1)} \leq a_i^{(j)}$  and  $b_i^{(j)} \leq b_i^{(j+1)}$ . Moreover, if in an infinite word  $s(T) = (\sigma^{(1)}, \sigma^{(2)}, \dots)$  holds  $\sigma_i^{(j+1)} = -1$ , then  $a_i^{(j+1)} = a_i^{(j)} - 2^{j+1}$ ,  $b_i^{(j+1)} = b_i^{(j)}$ . If  $\sigma_i^{(j+1)} = 1$ , then  $a_i^{(j+1)} = a_i^{(j)}$  and  $b_i^{(j+1)} = b_i^{(j)} + 2^{j+1}$ . Altogether we obtain

$$(3) \quad a_i^{(j+1)} = a_i^{(j)} + \frac{\sigma_i^{(j+1)} - 1}{2}(b_i^{(j)} - a_i^{(j)}),$$

$$(4) \quad b_i^{(j+1)} = b_i^{(j)} + \frac{\sigma_i^{(j+1)} + 1}{2}(b_i^{(j)} - a_i^{(j)}).$$

Note again, that from (3) and (4), in particular, it follows that the intersection of the tower  $W(T_j)$  with the horosphere  $E_0$  is a  $d$ -cube of edge length  $2^{j+1}$ .

**Theorem 4.5.** *In a B-tiling in  $\mathbb{H}^{d+1}$  the following properties hold:*

- (1) *The number of pools is  $2^k$  for some  $0 \leq k \leq d$ .*
- (2) *For any  $d$  and any  $0 \leq k \leq d$ , there are B-tilings in  $\mathbb{H}^{d+1}$  with  $2^k$  pools.*
- (3) *Given  $0 \leq k \leq d$ , in all B-tilings in  $\mathbb{H}^{d+1}$  with  $2^k$  pools the supports of all pools are pairwise congruent to each other.*
- (4) *Given a B-tiling  $T$ , all pools are pairwise congruent to each other with respect to tiles.*

(5) *All  $2^k$  pools in a B-tiling share a common  $(d - k + 1)$ -plane.*

*Proof.* Let  $\mathcal{P}$  be a pool in a B-tiling  $\mathcal{T}$ . By Proposition 4.4,  $\mathcal{P}$  is the union of towers  $(W(T_j))_{j \in \mathbb{N}}$ , where  $W(T_j) \subset W(T_{j+1})$ . Let  $E_0$  be a horosphere that contains some  $b$ -facet  $b \subset T \in \mathcal{P}$ . Since the intersection of a tower and  $E_0$  is a  $d$ -cube, the intersection of  $\mathcal{P}$  and  $E_0$  is the union of countably many  $d$ -cubes  $\{\square_j\}_{j \geq 0}$ , where  $\square_j \subset \square_{j+1}$ .

Let  $a_i^{(j)}, b_i^{(j)}$  as in (3),(4) correspond to  $\mathcal{P} = \bigcup_{j \geq 0} W(T_j)$ . From (3) and (4) we read off: if there are only finitely many  $k$ , such that  $\sigma_i^{(j)} = -1$ , then there is a sharp lower bound  $a$  for  $a_i^{(j)}$ . Therefore, in this case the union  $\bigcup_{j \geq 0} \square_j$  (where  $\square_j = W(T_j) \cap E_0$ ) is contained in the half-space  $H_i^+ = \{x = (x_1, \dots, x_d) \in E_0 = \mathbb{E}^d \mid x_i \geq a\}$ .

Analogously, if there are only finitely many  $j$ , such that  $\sigma_i^{(j)} = 1$ , then there is an upper bound  $b$  for  $b_i^{(j)}$ . In this case  $\bigcup_{j \geq 0} \square_j$  is contained in the half-space  $\tilde{H}_i^- = \{x = (x_1, \dots, x_d) \in \mathbb{E}^d \mid x_i \leq b\}$ . We obtain:

**(A)** For any fixed coordinate  $i$ ,  $\bigcup_{j \geq 0} \square_j$  is either unbounded in one direction or unbounded in both directions.

Assume a hyperplane  $\bar{h}$  is the boundary (at least partly) of a pool  $\mathcal{P}$  in  $\mathcal{T}$ . Consider a tile with a  $c$ -facet on  $\bar{h}$ . All tiles  $T_j \in t(T)$  have  $c$ -facets in the hyperplane  $\bar{h}$ . Now, since  $\mathcal{T}$  is face-to-face, the reflection  $\tau$  in the hyperplane  $\bar{h}$  moves  $T$  to some tile  $T'$  which shares with  $T$  a common  $c$ -facet. Note that, by assumption,  $T'$  belongs to another pool  $\mathcal{P}'$ . The tail  $t(T)$  moves under  $\tau$  into  $t(T')$ . Therefore, the pool  $\mathcal{P}$  moves under  $\tau$  into pool  $\mathcal{P}'$  too. And, in particular, the tail of  $T'$  — determining  $\mathcal{T} \cap \mathcal{P}'$  uniquely — is obtained from  $t(T)$  by a reflection. Thus  $s(T)$  differs from  $s(T')$  only by exchanging the sign in one coordinate.

Now, since  $\mathcal{T}$  is face-to-face, the hyperplane  $\bar{h}$ , which separates two pools, is a totally separating hyperplane. Indeed, if  $\bar{h}$  contains a common facet of two tiles in a horospheric layer, it cannot dissect any other tile in the same layer. Since, for being (part of) a boundary of two pools,  $\bar{h}$  separates some adjacent tiles in any layer, it cannot dissect a tile in tiling  $\mathcal{T}$  at all. Therefore, for two tiles  $T$  and  $T'$  lying on different sides of  $\bar{h}$ , their tails have no tiles in common.

Now, by (A), and since any hyperplane in the boundary of some pool is parallel to some  $c$ -facet, the hyperplanes in  $\partial\mathcal{P}$  are pairwise orthogonal to each other. Thus they partition  $\mathbb{H}^{d+1}$  into  $2^k$  pools, proving the first point of the theorem. It is a simple exercise to construct B-tilings with  $2^k$  pools for all  $0 \leq k \leq d$ . In fact, one can use sequences  $s(T)$ , where the numbers of 1s in exactly  $k$  coordinates is finite. This proves the second point of the theorem. Being pairwise orthogonal to each other, all the  $k$  hyperplanes share a common  $d - k + 1$ -dimensional plane which is also orthogonal to the horospheres, proving the fifth point of the theorem.

Now we know the structure of the intersection of the pool  $\mathcal{P}$  with the horosphere  $E$ :

$$\mathcal{P} \cap E = \mathbb{E}^m \oplus E^{+(d-m)},$$

where  $E^{+(d-m)}$  denotes a  $d - m$ -dimensional 'octant', i.e, the sum  $\mathbb{R}^+ \oplus \dots \oplus \mathbb{R}^+$  of  $d - m$  half-lines. Therefore, in all B-tilings with  $2^k$  pools, all supports of the pools are the same, which proves the third point of the theorem. Note that, up to here, two pools may have congruent supports, but can be pairwise different as tilings.

Now we prove that all  $2^k$  in some given B-tiling  $\mathcal{T}$  are pairwise congruent with respect to tiles. We did prove it already for two pools having some  $c$ -facet in common. But any two pools can be linked by a chain of pools in which all sequel pools have  $d$ -dimensional boundary in common. Therefore, the fourth point of the theorem is proved.  $\square$

Let us emphasise that point (4) of Theorem 4.5 implies that all tails in a given B-tiling are cofinally equivalent, up to multiplying entire coordinates (that is, entire sequences  $(\sigma_i^{(j)})_{j \in \mathbb{N}}$ ) by  $-1$ . This is stated precisely in the following corollary.

**Corollary 4.6.** *Let  $\mathcal{T}$  be a B-tiling, and let  $T, \tilde{T} \in \mathcal{T}$ . Denote their sequences by  $s(T) = (\sigma_1^{(j)}, \dots, \sigma_d^{(j)})_{j \in \mathbb{N}}$ ,  $s(\tilde{T}) = (\tilde{\sigma}_1^{(j)}, \dots, \tilde{\sigma}_d^{(j)})_{j \in \mathbb{N}}$ . Then there are  $m, n \in \mathbb{N}$  such that for each  $i \leq d$  holds*

$$\forall j \in \mathbb{N} : \sigma_i^{(j+m)} = \tilde{\sigma}_i^{(j+n)} \quad \text{or} \quad \forall j \in \mathbb{N} : \sigma_i^{(j+m)} = -\tilde{\sigma}_i^{(j+n)}.$$

*If  $T$  and  $\tilde{T}$  are contained in the same layer, then there is  $m \in \mathbb{N}$  such that*

$$\forall j \geq m : \sigma_i^{(j)} = \tilde{\sigma}_i^{(j)} \quad \text{or} \quad \forall j \geq m : \sigma_i^{(j)} = -\tilde{\sigma}_i^{(j)}.$$

*Proof.* By Theorem 4.5, all pools in  $\mathcal{T}$  are congruent with respect to tiles. This was proven by the fact, that any two pools are mapped to each other by reflections  $\tau_i$  in hyperplanes supporting the boundary of the pools. This means that all tails are cofinally equivalent after applying some of the reflections  $\tau_i$ . These reflections act as multiplication by  $-1$  in the  $i$ -th coordinate, and the first claim follows. The second claim covers the special case where the tiles are contained in the same layer. Then the tails coincide from some common position  $m$  on, up to reflections  $\tau_i$ .  $\square$

## 5. Symmetries of Böröczky Tilings in $\mathbb{H}^{d+1}$

In the last section we strongly used the fact that horospheres in  $\mathbb{H}^{d+1}$  are isometric to  $\mathbb{E}^d$ , compare for instance [11, §4.7]. This will also be useful in the sequel, where we apply the results of the last section to determine all possible symmetries of a B-tiling.

**Theorem 5.1.** *Let  $\mathcal{T}$  be a Böröczky tiling in  $\mathbb{H}^{d+1}$  with  $2^k$  pools, let  $0 \leq k \leq d$ , and denote by  $\text{Sym}(\mathcal{T})$  its symmetry group. Then  $\text{Sym}(\mathcal{T})$  is isomorphic to*

- $\mathbb{Z} \times B_k$  if there is a periodic sequence in  $\mathcal{T}$ , or
- $B_k$  else;

where  $B_k$  is the symmetry group of a  $k$ -cube.

The notation  $B_k$  follows Coxeter, see [7].  $B_k$  is the group with Coxeter diagram



*Proof.* Let  $S$  be the union of all  $k$  hyperplanes bounding the  $2^k$  pools in  $\mathcal{T}$ . In the following,  $\varphi$  denotes always a symmetry of  $\mathcal{T}$ , i.e., an isometry of  $\mathbb{H}^{d+1}$  with the property  $\varphi(\mathcal{T}) = \mathcal{T}$ . Then, in particular,  $\varphi(S) = S$ .

The proof is organised as follows: First we prove that a group isomorphic to  $B_k$  is contained in  $\text{Sym}(\mathcal{T})$ . Then we show that there is  $\varphi \in \text{Sym}(\mathcal{T})$  which maps some horosphere  $E_0$  to some horosphere  $E_j \neq E_0$  if and only if there is a tail which is cofinally periodic (Claims 1,2,3).

Finally it is shown that any  $\varphi \in \text{Sym}(\mathcal{T})$  fixing some horosphere  $E_0$  is element of  $\bar{B}_k$  (Claims 4,5,6).

By Theorem 4.5, the intersection of the  $k$  hyperplanes in  $S$  is a plane  $H^{d-k+1}$  which is orthogonal to the horospheres  $E_j$  in  $\mathbb{H}^{d+1}$ . The intersection of  $S \cap E_j$  for any  $j$  consists of  $k$  pairwise orthogonal hyperplanes  $h_1 \dots, h_k$  in  $E_j = \mathbb{E}^d$ . They share a common  $d - k$ -dimensional Euclidean plane.

Since a reflection  $\tau_i$  in any of these  $k$  hyperplanes, as we have seen, keeps the tiling  $\mathcal{T}$  invariant, the group generated by the reflections  $\langle \tau_i \rangle \subseteq \text{Sym}(\mathcal{T})$ .

The restrictions  $\nu_i$  of the hyperbolic reflections  $\tau_i$  onto the  $E$  are Euclidean reflections of  $E$  in hyperplanes  $h_i$ . Besides these reflections there are reflections  $\nu_{ij}$  in bisectors of all dihedral angles between hyperplanes  $h_i$  and  $h_j$ , which also keep the set  $S$  invariant. A Coxeter group generated by all  $\nu_i$  and  $\nu_{ij}$ ,  $1 \leq i, j \leq k$ , is exactly a group  $B_k$  of the  $k$ -dimensional cube. Denote by  $\bar{B}_k$  a Coxeter group  $\langle \tau_i, \tau_{ij} \rangle$  generated by corresponding reflections of  $\mathbb{H}^{d+1}$  and show that  $\bar{B}_k \subseteq \text{Sym}(\mathcal{T})$ .

Let  $\mathcal{P} \subseteq \bar{h}_i^+ \cap \bar{h}_j^+$  and  $\bar{h}_{ij}$  the bisector of the dihedral angle  $\angle \bar{h}_i \bar{h}_j$ . Let us consider a tile  $T \in \mathcal{P}$  which is 'inscribed' into the  $\angle \bar{h}_i \bar{h}_j$ , that is, two  $c$ -facets of  $T$  lie on the boundary hyperplanes  $\bar{h}_i$  and  $\bar{h}_j$ . The bisector  $\bar{h}_{ij}$  dissects horospheric  $a$ - and  $b$ -facets of  $T$ , which are  $d$ -dimensional Euclidean cubes, into two parts. It is clear that the reflection  $\tau_{ij}$  moves  $T$  into itself. So,  $\tau_{ij}(T) = T$  and, consequently,  $\tau_{ij}(t(T)) = t(T)$  and  $\tau_{ij}(\mathcal{P}) = \mathcal{P}$ . By the face-to-face property, this implies that  $\tau_{ij}(\mathcal{T}) = \mathcal{T}$ . Therefore,  $\bar{B}_k \subseteq \text{Sym}(\mathcal{T})$ .

We proceed by showing that there are no other symmetries, except possibly the ones arising from shifts along some line orthogonal to the horospheres  $E_i$  (possibly followed by some  $\nu \in \bar{B}_k$ ). These symmetries correspond to the occurrence of the infinite cyclic group.

*Claim 1:*  $\varphi(t(T)) = t(\varphi(T))$ .

The set  $\varphi(t(T)) = \{\varphi(T), \varphi(T_1), \varphi(T_2), \dots\}$  is clearly a tail. Since, for each fixed B-tiling  $\mathcal{T}$ , each tail is uniquely determined by its first element, the claim follows.

*Claim 2:* If  $T$  and  $\varphi(T)$  are in different layers, then  $s(T)$  is cofinally periodic.

Let  $T$  and  $\tilde{T} = \varphi(T)$  be in different layers. Then, by Corollary 4.6, there are  $m, n \in \mathbb{N}$  such that for their sequences holds:

$$(5) \quad \forall j \in \mathbb{N} : \sigma_i^{(j+m)} = \tilde{\sigma}_i^{(j+n)} \quad \text{or} \quad \forall j \in \mathbb{N} : \sigma_i^{(j+m)} = -\tilde{\sigma}_i^{(j+n)} \quad (1 \leq i \leq d).$$

Since  $T$  and  $\tilde{T}$  are in different layers, we have  $m \neq n$ . Without loss of generality, let  $m > n$ . By Claim 1,  $t(T)$  and  $t(\tilde{T})$  are congruent. Therefore  $s(T)$  and  $s(\varphi(T))$  are identical, up to multiplication of entire sequences  $(\sigma_i^{(j)})_{j \in \mathbb{N}}$  by  $-1$ . Thus, for each  $1 \leq i \leq d$ ,

$$(6) \quad \forall j \in \mathbb{N} : \sigma_i^{(j)} = \tilde{\sigma}_i^{(j)} \quad \text{or} \quad \sigma_i^{(j)} = -\tilde{\sigma}_i^{(j)}$$

Let  $k = m - n$ . From (5) and (6) follows for each  $1 \leq i \leq d$ :

$$\forall j \in \mathbb{N} : \sigma_i^{(j+k)} = \pm \tilde{\sigma}_i^{(j)} = \begin{cases} \pm \sigma_i^{(j)} \\ \mp \sigma_i^{(j)} \end{cases}$$

We obtain either  $\sigma_i^{(j+k)} = \sigma_i^{(j)}$ , or  $\sigma_i^{(j+k)} = -\sigma_i^{(j)}$ . In the second case holds for  $j \geq k$ :  $\sigma_i^{(j+k)} = \sigma_i^{(j-k)}$ . In each case,  $p_i = 2k$  is a period of  $\sigma_i$ . Then, the lowest common multiple

of all the  $p_i$  is a period of  $s(T)$ . So we can already conclude: If the sequence of one pool is not periodic, any possible symmetry of  $\mathcal{T}$  maps tiles onto tiles in the same layer.

*Claim 3:* If there is a cofinally periodic sequence  $s(T)$  for some  $T \in \mathcal{T}$ , then there exists a symmetry  $\varphi$  such that  $\varphi^k(\mathcal{T}) = \mathcal{T}$  for all  $j \in \mathbb{Z}$ .

Let  $s(T)$  be a periodic sequence in  $\mathcal{T}$ . Then there is a tile  $T$  with a tail  $t(T)$  which belongs to this sequence. This tail is infinite in one direction. In the other direction we can extend it in any way we want (since by passing from  $T$  through a  $b$ -facet to another tile, we can choose each of the  $2^d$  tiles which are lying there). In particular, we can extend  $t(T)$  to a biinfinite horospheric path  $t_b = (\dots, T_{-2}, T_{-1}, T_0 = T, T_1, T_2, \dots)$ , which belongs to a biinfinite periodic sequence  $s_b = (\dots, \sigma^{(-2)}, \sigma^{(-1)}, \sigma^{(0)}, \sigma^{(1)}, \sigma^{(2)}, \dots)$ . Let  $k$  be the period of  $s_b$ , then it holds

$$\dots = \sigma^{(i-2k)} = \sigma^{(i-k)} = \sigma^{(i)} = \sigma^{(i+k)} = \sigma^{(i+2k)} = \dots$$

for every  $i \in \mathbb{Z}$ . So for every  $i \in \mathbb{Z}$ , the set  $t^{(i)} = \{T_i, T_{i+1}, T_{i+2}, \dots\}$  is congruent to  $t^{(i+k)} = \{T_{i+k}, T_{i+k+1}, T_{i+k+2}, \dots\}$ . In other words, there is an isometry  $\varphi$  such that  $\varphi(t^{(i)}) = t^{(i+k)}$ . Consequently,  $\varphi(t_b) = t_b$ , and therefore  $\varphi^j(t_b) = t_b$  for every  $j \in \mathbb{Z}$ . From Proposition 4.4 follows that  $t_b$  determines its pool uniquely. By Theorem 4.5, this pool determines the whole tiling. It follows  $\varphi^j(\mathcal{T}) = \mathcal{T}$  for  $j \in \mathbb{Z}$ .

The deduced symmetry  $\varphi$  is obviously a shift along some line  $\ell$ . If the period  $k$  is not prime, it is possible that there is an essential period (see the end of Section 2) smaller than  $k$ . However, some power of some symmetry  $\psi$  corresponding to such an essential period is a shift along a line:  $\psi^m = \varphi$ , with  $\varphi$  as above.

By considering the action of  $\varphi$  on tails  $t(T)$ , it follows that  $\varphi$  maps the horospheres  $E_i$  to  $E_{i+k}$ . Hence  $\varphi^m \neq \varphi^j$  for  $m \neq j$ . This gives rise to the occurrence of an infinite cyclic group in the symmetry group of  $\mathcal{T}$ .

It remains to consider symmetries which fix some (and thus each) horosphere  $E_i$  in the sequel. This is the same as requiring  $\varphi$  to fix some layer  $\mathcal{R}$ :  $\varphi$  then fixes also the boundary  $\partial\mathcal{R} = E_i \cup E_{i+1}$ . Since  $\varphi$  maps  $a$ -facets to  $a$ -facets and  $b$ -facets to  $b$ -facets, it fixes  $E_i$  as well as  $E_{i+1}$ .

*Claim 4:* Let  $T \in \mathcal{T}$ . Every symmetry  $\varphi$ , where  $\varphi(T)$  and  $T$  are in the same layer  $\mathcal{R}$  and in the same pool, has a fixed point in  $\mathcal{R}$ .

By Claim 1,  $\varphi$  maps  $t(T)$  to  $\varphi(t(T)) = t(\varphi(T))$ . Since  $T$  and  $\varphi(T)$  are in the same pool and in the same layer, these tails coincide from some position  $k$  on. In particular, there is  $T_k \in t(T)$  such that  $\varphi(T_k) = T_k$ . Thus, by Brouwer's fixed point theorem,  $\varphi$  fixes some point in  $T_k$ . Moreover, by the symmetry of  $T_k$ ,  $\varphi$  fixes some point  $x_a$  in the  $a$ -facet of  $T_k$ , as well as some point  $x_b$  in some  $b$ -facet of  $T_k$ . Being an isometry,  $\varphi$  fixes the line  $\ell'$  through  $x_a$  and  $x_b$  pointwise. Therefore, the intersection  $\ell' \cap \mathcal{R}$  consists of fixed points of  $\varphi$ .

In fact, these symmetries are those arising from the reflections  $\tau_{ij}$ .

*Claim 5:* Let  $T \in \mathcal{T}$ . Every symmetry  $\varphi$ , where  $\varphi(T)$  and  $T$  are in the same layer  $\mathcal{R}$  but in different pools  $\mathcal{P}, \mathcal{P}'$ , has a fixed point in this layer.

This can be shown in analogy to the proof of the last claim. By Theorem 4.5,  $\mathcal{P} \cap \mathcal{P}'$  intersect in a common  $(d-k+1)$ -plane. Thus there are tiles  $\tilde{T} \in \mathcal{P}$ ,  $\tilde{T}' \in \mathcal{P}'$  having  $d-k+1$ -dimensional intersection. By the face-to-face property, this intersection is a  $(d-k+1)$ -face  $F$ . By the proof of Theorem 4.5, this is also true for any pair of tiles  $\tilde{T}_k \in t(\tilde{T}), \tilde{T}'_k \in t(\tilde{T}')$ , where  $k \in \mathbb{N}$ . By Proposition 4.4, there is  $k$  such that also holds:  $\tilde{T}_k \in t(T), \tilde{T}'_k \in t(\varphi(T))$ . Now, similar as

in the proof of the last claim,  $\varphi$  fixes a  $(d - k + 1)$ -face  $F' = \tilde{T}_k \cap \tilde{T}'_k$ . Moreover,  $\varphi$  fixes some point  $x_a \in F$  ( $x_b \in F$ ), contained in the intersection of  $F$  with the  $a$ -facets ( $b$ -facets) of  $\tilde{T}_k$  and  $\tilde{T}'_k$ . As above,  $\varphi$  fixes the line  $\ell'$  through  $x_a$  and  $x_b$  pointwise. Therefore, the intersection  $\ell' \cap R$  consists of fixed points of  $\varphi$ .

In fact, these symmetries are the ones corresponding to the reflections  $\tau_i$ .

*Claim 6:* Let  $\varphi \in \text{Sym}(\mathcal{T})$  fix some (and thus each) horosphere  $E_i$ . Then  $\varphi \in \bar{B}_k$ .

By the construction of a B-tiling,  $\mathcal{T}$  induces a cube tiling of  $E_i$  by  $a$ -facets. For any symmetry  $\varphi$  which fixes both  $\mathcal{T}$  and  $E_i$ , the restriction  $\nu := \varphi|_{E_i}$  fixes this cube tiling. By the last two claims, each such  $\nu$  has a fixed point in  $E_i$ . In particular,  $\nu$  is not a translation. But, fixing a cube tiling in  $E_i = \mathbb{E}^d$ , the set of all these  $\nu$  form a crystallographic group. The crystallographic groups which fix some cube tiling and which contain no translations are well known, see for instance [7]. These are exactly the subgroups of  $B_k$ , which proves Claim 6.

Altogether, we found two kinds of possible symmetries: Symmetries in  $\bar{B}_k$ , and shifts along a line orthogonal to each  $E_i$ , possibly followed by some map  $\tau \in \bar{B}_k$ .  $\square$

A consequence of Theorem 5.1 is that all Böröczky-type tilings are non-crystallographic. This can also be shown by Theorem 3.5, along the same lines as in Section 3 for the 2-dimensional tiling: It is not hard to convince oneself that the number of  $k$ -coronae in a Böröczky-type tiling in  $\mathbb{H}^{d+1}$  for  $d > 2$  is strictly larger than  $2^{k-1}$  for  $k \geq 2$ , but a proper proof may be lengthy, and yields no new result.

As another consequence of Theorem 5.1 we obtain:

**Corollary 5.2.** *Almost every Böröczky-type tiling has finite symmetry group.*

*Proof.* Theorem 5.1 shows, that  $\text{Sym}(\mathcal{T})$  is infinite, if and only if there is a tail  $t(\mathcal{T})$  with a periodic sequence  $s(\mathcal{T})$ . Since there are only countably many of these, there are only countably many congruence classes of Böröczky-type tilings in  $\mathbb{H}^{d+1}$  with an infinite symmetry group. Since the Böröczky-type tilings are non-crystallographic, it follows from [5] that there are uncountably many congruence classes of them, which proves the claim.  $\square$

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#### REFERENCES

- [1] J.W. Anderson, *Hyperbolic Geometry*, Springer, New York (1999).
- [2] K. Böröczky: Gömbkitöltések állandó görbületű terekben I, II, *Mat. Lapok*, **25**, 265-306, **26**, 67-90 (1974).
- [3] B.N. Delone, N.P. Dolbilin, M.I. Stogrin, R.V. Galiulin: A local criterion for regularity of a system of points, *Soviet Math. Dokl.*, **17**, Vol. 2, 319-322 (1976).
- [4] N.P. Dolbilin: Which clusters can form a crystal? in: *Voronoi's impact on modern science. Book II*. eds: P. Engel et al., Proc. Inst. Math. Natl. Acad. Sci. Ukr., Math. Appl. **21** (2), pp. 96-104 (1998).
- [5] N.P. Dolbilin: The countability of a tiling family and the periodicity of a tiling, *Discrete Comput. Geom.* **13**, 405-414 (1995).
- [6] G. Fejes Tóth, W. Kuperberg: Packing and Covering with Convex Sets, *Handbook of Convex Geometry*, Vol. B, ed. P.M. Gruber, J.M. Wills, North-Holland, Amsterdam, pp. 861-898 (1993).

- [7] J.E. Humphreys: *Reflection Groups and Coxeter Groups*, Cambridge University Press (1990).
- [8] V.S. Makarov: On a non-regular partition of an  $n$ -dimensional Lobachevskii space by congruent polytopes (Russian), *Trudy Mat. Inst. Steklov* **196** (1991), translated in: Proc. Steklov Inst. Math. 1992 no. 4, 103-106.
- [9] S. Mozes: Aperiodic Tilings, *Inv. Math.* **128** (1997) 603-611.
- [10] R. Penrose: Pentaplexity. A class of non-periodic tilings of the plane, *Math. Intell.* **2**, (1979) 32-37.
- [11] J.G./ Ratcliffe, *Foundations of Hyperbolic Manifolds*, Springer, New York (1994).
- [12] È.B. Vinberg, O.V. Shvartsman: Discrete groups of motions of spaces of constant curvature, Geometry, II, Encyclopaedia Math. Sci. **29**, Springer, Berlin (1993) pp. 139-248.

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